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Default clearing and ex-ante contagion in financial systems with a two-layer network structure

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Abstract

Systemic risks do not arise only as a result of a crisis event, and it is important to understand the ex-ante risk contagion mechanisms. There has been no research on ex-ante contagion valuation and contagion modelling of multilayer networks. This study derives the ex-ante-contagion mechanism of a two-layer network financial system with interbank lending connections and cross-holding connections, constructs a general valuation model of the financial system based on the Eisenberg and Noe clearing framework, and then obtains a model of ex-ante risk contagion and valuation functions. It is further verified by stress tests that bankruptcy is not a necessary condition for loss generation. By simulating different shock scenarios, we obtain the systemic risk and systemically important banks in China. Our models and analyses provide new research perspectives for studying risk contagion mechanisms in financial networks and provide empirical corroboration for regulators and policy makers.

Keywords: Two-layer networks, Risk contagion, Default clearing, Ex-ante valuation, Systemic risk

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1. Introduction

The outbreak of the global financial crisis has raised continued focus among scholars on the risks of the financial system. In the context of the current complex economic environment, how to effectively prevent and resolve systemic financial risks induced by economic fluctuations and market shocks has become a key concern for financial regulators and also academics. Banking financial institutions are the priority for systemic financial risk prevention, and most of the risks in the financial system are concentrated within the banking system [**?**]. One of the crucial causes of financial crisis after bank bankruptcy is interbank business connections. In the existing systemic risk research, constructing complex networks based on interbank connections is a hot research topic [**? ? ?**]. Recent studies construct network topologies through two ways: network topology analysis based on interagency business or market data [**? ? ?**] and simulation-based modelling of network topologies [**? ?**]. However, most of these studies have designed network topologies based on a particular type of business or market connection. Banks can establish internal connections through multiple identities. Under the interaction of different connections, banks form contagion pathways characterized by interdependence and influence, exhibiting complex network characteristics with multi-level and multi-link interactions [**?**]. Considering the contagion mechanism only with a single type of connection does not fully reflect the real financial system, which may lead to an underestimation of systemic risk [**?**].

Multi-layer networks provide a more accurate representation of the interconnections between banks. **?**] constructed a multilayer network model considering both liabilities and cross-holdings and found that multilayer financial networks have a nonlinear effect in risk contagion. **?**] extended the DebtRank model to multi-channel contagion of interbank lending, cross-holdings and overlapping investments, and the empirical results verified that considering only a single channel underestimates the systemic risk. **?**] analyzed the contagion mechanisms resulting from the combined effects of four contagion channels: financing contagion, overlapping portfolio contagion, counterparty risk contagion and deleveraging contagion, and found that the instability caused by the interacting channels may be much greater than the sum of the individual channels acting alone. Due to the limitation of data availability, some studies have constructed multilayer networks through stock price similarity or modelled multilayer network structures based on balance sheet indicators [? ? ?].

Contagion is an important feature of systemic risk. In researches of risk contagion mechanisms, some works argue that risk contagion occurs only when a bank experiences an actual solvency crisis or default, i.e., systemic risk is created through ex-post contagion after a localized crisis has occurred [? ?]. Banks do not estimate the probability that their counterparties will default, but only observe whether they have defaulted. ?] analyzed interbank lending and investigated the existence and uniqueness of payment vectors in this system. This framework (also called the EN clearing framework) has been widely used in research since then. ?] added explicit bankruptcy costs to EN model, ?] considers bank equity connections, and ?] allows for bank defaults on arbitrarily senior debt structures. Besides clearing approach, some default cascade models also assume that risk contagion occurs only if a bank defaults, such as those by [?] and [?].

In reality, it is extremely unlikely that a bank default will occur. Under the mark-to-market (MTM) framework, banks continuously adjust their interbank lending of counterparties to market value. Generally, when a counterparty incurs losses, it is often accompanied by a deterioration in creditworthiness, leading to a decline in the market value of interbank assets. This results in contagion losses. Thus, systemic risk can still arise through credit valuation adjustment (CVA) even if no bank defaults [?]. According to the estimates of the Basel Committee on Banking Supervision (BCBS), most of the losses during the 2008 financial crisis were not due to bank defaults but were instead caused by MTM reductions stemming from [?]. Therefore, ex-ante contagion of financial risks deserves more attention than ex-post contagion. There are already some studies on ex-ante contagion mechanisms. ?] proposed the DebtRank default cascade model in which the default probability is viewed as a linear function of relative equity losses, using simple heuristic rules for the ex-ante valuation of bank equity. Building on this, ?] improve the default probability as a nonlinear function of the loss of equity. ?] and ?] introduced the ?] classical credit risk structure model to calculate default probability in the EN clearing model, which allowed ex-ante contagion to occur.

Research on the financial systemic risk contagion mechanism has yielded abundant results, but there are still issues that require further investigation: (i) the above studies still focus on "ex-post" contagion and do not take into account the spot losses arising from future defaults of counterparties; (ii) some of the literature has constructed models of pre-default risk contagion of banks, but they only consider the contagion of risk through a single channel of interbank lending, and no scholars have yet conducted studies on multi-channel ex-ante contagion models under the clearing framework. There is no research on multi-channel ex-ante contagion models in the clearing framework. Addressing the above issues, our study constructs a two-layer interbank business connection network based on interbank lending and cross-holding. Within the EN clearing framework, a two-channel ex-ante valuation model for financial systems is developed to investigate the ex-ante contagion characteristics of systemic risk.

Our contributions are fourfold: (1) We refine the setup of contagion channels by constructing a two-layer network banking system based on interbank lending and cross-shareholding. (2) We incorporate cross-holding as a contagion channel into the ex-ante valuation model to more fully reflect the internal connections of the banking system. (3) We construct a valuation model based on the two-channel contagion path, on the basis of which we construct a feasible ex-ante valuation model.

This paper is organized as follows. In Section **??** we present the two-layer network ex-ante contagion model. In Section **??** we perform a stress test and analyze the results. Some conclusions are reported in Section **??**.

2. Model

2.1. A two-layer networked financial system

Consider a two-layer network financial system consisting of N banks, $\mathcal{N} = \{1, 2, \ldots, N\}$, and the system holds M external assets, $\mathcal{M} = \{1, 2, \ldots, M\}$. Bank assets consist of three components, external assets, interbank lending assets, and interbank cross-holding assets. Bank liabilities consist of external liabilities and interbank liabilities, and each of these components can be represented as a matrix. We assume that all liabilities have the same claim priority at maturity. The concepts of the financial system are summarized in Definition **??**.

Definition 2.1 (Financial System). *A financial system $\mathcal{B}$ is given by the tuple $(\mathbf{A^c}, \mathbf{A^f}, \mathbf{A^s}, \mathbf{L})$, where*

(1) $\mathbf{A^c} \in \mathbb{R}^{N \times M}$ and $\mathbf{L^c}$ are the external asset matrix and external liability matrix, respectively, for $\forall i \in \mathcal{N}$, $k \in \mathcal{M}$, $A^c_{ik} \geq 0$, $L^c_i \geq 0$;

(2) $\mathbf{A^f} \in \mathbb{R}^{N\times N}$ *is an interbank lending asset matrix with* $A^f_{ij} \geq 0$ *for* $\forall i, j \in \mathcal{N}$ *and* $\forall i$, $A^f_{ii} = 0$. *The interbank liability matrix* $\mathbf{L^f} = \left(\mathbf{A^f}\right)'$, *i.e.,* $A^f_{ij} = L^f_{ji}$;

(3)$\mathbf{A^s} \in \mathbb{R}^{N\times N}$ *is a cross-holding asset matrix with* $A^s_{ij} \geq 0$ *for* $\forall i, j \in \mathcal{N}$ *and* $\forall i, A^s_{ii} = 0$;

(4) $\mathbf{E}$ *is the equity vector.*

Let $\overline{\mathbf{A}}^{\mathbf{c}} = [\mathbf{A^c}]\,\mathbf{1}$, $\overline{\mathbf{A}}^{\mathbf{f}} = \left[\mathbf{A^f}\right]\mathbf{1}$, and $\overline{\mathbf{A}}^{\mathbf{s}} = [\mathbf{A^s}]\,\mathbf{1}$ denote the total external asset vector, the total interbank lending asset vector, and the total cross-holding asset vector respectively. $\overline{\mathbf{L}}^c$ and $\overline{\mathbf{L}}^f$ denote the total external liabilities vector and the total interbank liabilities vector, respectively, and the total liabilities $\overline{\mathbf{L}} = \overline{\mathbf{L}}^c + \overline{\mathbf{L}}^f$. According to the accounting equation, we have $\mathbf{E} = \overline{\mathbf{A}}^{\mathbf{c}} + \overline{\mathbf{A}}^{\mathbf{f}} + \overline{\mathbf{A}}^{\mathbf{s}} - \overline{\mathbf{L}}$.

Simplify the formula to make the subsequent article easier to understand and define the following symbols:

Total external assets: $\overline{A}^c_i = \sum_{k=1}^M A^c_{ik} = \sum_{k=1}^M c_{ik}\overline{C}_k$.

Total interbank lending assets: $\overline{A}^f_i = \sum_{j=1}^N A^f_{ij} = \sum_{j=1}^N f_{ij}\left(\overline{L}^c_j + \overline{L}^f_j\right) = \sum_{j=1}^N f_{ij}\overline{L}_j$.

Total cross-holding assets: $\overline{A}^s_i = \sum_{j=1}^N A^s_{ij} = \sum_{j=1}^N s_{ij}\overline{S}_j$.

The ratio c_{ik} represents the proportion of k assets held by bank i to the value of k assets $\overline{C}_k$. f_{ij} is the proportion of bank i's interbank lending assets held by j to j's total debt $\overline{L}_j$. s_{ij} is the proportion of the value of i's equity holding in j to the value of j's total equity $\overline{S}_j$.

Remark 2.1. *The ratios* f_{ij}, s_{ij} *satisfy*

$$0 \leq \sum_{i=1}^N f_{ij} \leq 1,\ 0 \leq \sum_{i=1}^N s_{ij} \leq 1,\ f_{jj} = s_{jj} = 0,\ \forall i, j \in \mathcal{N}, \tag{1}$$

and there exists at least one i *such that*

$$\sum_{i=1}^N f_{ij} < 1 \ \mathit{or} \sum_{i=1}^N s_{ij} < 1. \tag{2}$$

Remark **??** implies that there is at least one debt holder or equity holder of a bank outside the system.

Table **??** illustrates the initial balance sheet of bank i.

Table 1: Initial balance sheet of bank i

Assets	Liabilities
External asset$\overline{A}_i^c$	External liability$\overline{L}_i^c$
Interbank lending asset$\overline{A}_i^f$	Interbank liability$\overline{L}_i^f$
Cross-holding asset $\overline{A}_i^s$	Equity$\overline{E}_i$

The financial system constructed in our study has two forms of interbank connections: interbank lending connections and cross-holding connections, and there are two layers of risk contagion channels in this framework. Combining the two contagion channels, the financial system is effectively linked into a multi-layered network structure. As shown in Fig. **??**, risk can be transmitted both in the interbank lending layer $\mathbf{A^c}$ and the cross-holding layer $\mathbf{A^f}$, as well as across these layers. In the end, we introduce an aggregation layer $\mathbf{A^{agg}}$ to represent the total contagion path.

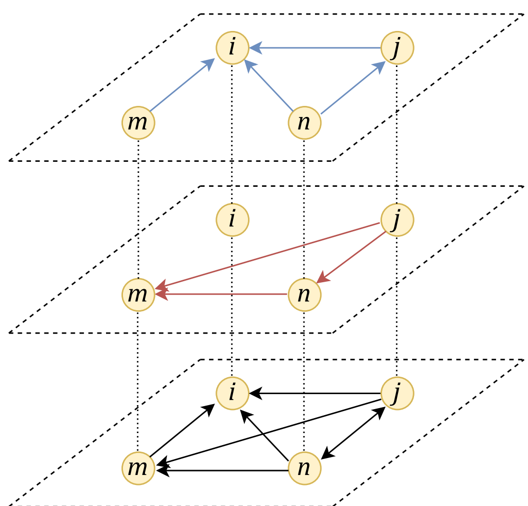


Figure 1: The representation of a two-layer banking network

2.2. Default and Clearing equilibrium

Most studies based on the EN clearing framework [**?**] define default by checking whether a certain value is less than the total liabilities. Based on previous studies, we give general definitions about default and clearing payments.

Definition 2.2 (Default). *A* **??** $\left(\mathbf{A^c}, \mathbf{A^f}, \mathbf{A^s}, \mathbf{L}\right)$ *consisting of banks* $\mathcal{N}$ *with a total liability* $\overline{\mathbf{L}}$*. Define the default bank set* $\mathcal{D}$

$$\mathcal{D} := \{i \in \mathcal{N} \mid \overline{A}_i^c + \overline{A}_i^f + \overline{A}_i^s < \overline{L}_i\}. \tag{3}$$

That is, a bank defaults when its assets are unable to repay its total liabilities. Definition **??** can be obtained from the balance sheet and the accounting equation.

In the general case, each bank in the system is able to fully repay its debts. At this time, bank i will pay off its debt $\overline{L}_i$, and the remaining equity after debt repayment $\left(\overline{A}_i^c + \overline{A}_i^f + \overline{A}_i^s - \overline{L}_i\right)$ will be proportionally distributed to its shareholders. However, when one or more banks in the **??** are poorly managed or affected by external shocks to asset prices, leading to a reduction in bank assets and an inability to repay debts, the actual debt repayments to other banks will inevitably be lower than the amounts due. We use $\overline{\mathbf{L}}^*$ to denote the actual repayment of interbank debt at maturity, $\overline{L}_{ij}^* \in \left[0, \overline{L}_{ij}\right]$, and $\overline{\mathbf{S}}^*$ to denote the actual payment of bank equity, $\overline{S}_{ij}^* \in [0, +\infty)$.

Definition 2.3 (Clearing payment rule). *The clearing payment* $(\overline{\mathbf{L}}^*, \overline{\mathbf{S}}^*)$ *of the system* $\left(\mathbf{A^c}, \mathbf{A^f}, \mathbf{A^s}, \mathbf{L}\right)$ *follows three principles of bankruptcy law [***? ? ?** *]:*

(1) Limited Liability: Bank is not required to make any payment in excess of the its total clearing assets.

(2) Claims Priority: At maturity, shareholders will not be entitled to any value of the defaulting bank as long as there are any outstanding bonds owned to any creditor. Equity is the residual claim.

(3) Proportional Payments: At maturity, banks typically distribute clearing payments proportionally among their counterparties.

According to Definition **??**, the relative proportions of debt and equity paid by node j to node i are f_{ij} and s_{ij}. At maturity date T, the bank is cleared according to the clearing rule. Let the clearing payment vector $\mathbf{Z}(T) = \left(\overline{\mathbf{L}}^*, \overline{\mathbf{S}}^*\right)'$, and let

$$\mathbf{c} = \begin{pmatrix} c_{11} & \cdots & c_{1M} \\ \vdots & \ddots & \vdots \\ c_{N1} & \cdots & c_{NM} \end{pmatrix}, \mathbf{z} = \begin{pmatrix} f_{11} & \cdots & f_{1N} & s_{11} & \cdots & s_{1N} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{N1} & \cdots & f_{NN} & s_{N1} & \cdots & s_{NN} \end{pmatrix}.$$

According to the Definition **??**, there are

$$\overline{\mathbf{L}}^* = \min\left\{\mathbf{c}\overline{\mathbf{C}}(T) + \mathbf{z}\mathbf{Z}(T), \overline{\mathbf{L}}\right\}, \tag{4}$$

$$\overline{\mathbf{S}}^{*} = \max\left\{\mathbf{c}\overline{\mathbf{C}}\left(T\right) + \mathbf{z}\mathbf{Z}\left(T\right) - \overline{\mathbf{L}}, 0\right\}, \tag{5}$$

then

$$\mathbf{Z}\left(T\right) = \left(\overline{\mathbf{L}}^{*}, \overline{\mathbf{S}}^{*}\right)' = \begin{pmatrix} \min\left\{\mathbf{c}\overline{\mathbf{C}}\left(T\right) + \mathbf{z}\mathbf{Z}\left(T\right), \overline{\mathbf{L}}\right\} \\ \max\left\{\mathbf{c}\overline{\mathbf{C}}\left(T\right) + \mathbf{z}\mathbf{Z}\left(T\right) - \overline{\mathbf{L}}, 0\right\} \end{pmatrix} = \Phi\left(\mathbf{Z}\left(T\right)\right). \tag{6}$$

Clearly, Equation (**??**) represents a fixed-point problem concerning $\mathbf{Z}^{*}\left(T\right)$. The solution to this fixed-point problem is referred to as the "no-arbitrage equilibrium clearing payment", under which there will be no possibility of capital structure arbitrage.

Proposition 2.1. *Let $\mathbb{R}^{k}_{[0,\infty]} = \left\{\left(x_1, \ldots, x_k\right)^{t}, x_1, \ldots, x_k \in [0, \infty)\right\}$ denote the k-dimensional non-negative real space. In a system with N banks, the function Φ has a unique fixed point $\mathbf{Z}^{*}\left(T\right)$. For all $\mathbf{Z_0}\left(T\right) \in \mathbb{R}^{2N}_{[0,\infty)}$, the equilibrium clearing payment vector $\mathbf{Z}^{*}\left(T\right)$ can be expressed as*

$$\mathbf{Z}^{*}\left(T\right) = \lim_{n\to\infty} \Phi^{n}\left(\mathbf{Z_0}(T)\right) = \lim_{n\to\infty} \underbrace{\Phi \circ \cdots \circ \Phi}_{n}\left(\mathbf{Z_0}(T)\right). \tag{7}$$

The proof of Proposition **??** is in Appendix **??**.

2.3. Valuation model for counterparty risk

At maturity, according to the accounting equation, there is $\overline{A}^{c}_{i}\left(T\right) + \overline{A}^{f}_{i}\left(T\right) + \overline{A}^{s}_{i}\left(T\right) = \overline{L}^{c}_{i}\left(T\right) + \overline{L}^{f}_{i}\left(T\right) + \overline{E}_{i}\left(T\right)$. In the equation, interbank lending assets and interbank liabilities are at face value. Therefore, $\overline{E}_{i}\left(T\right)$ and $\overline{A}^{s}_{i}\left(T\right)$ are the equity and cross-holding assets, respectively, under the assumption that all counterparties will fulfill their obligations and that bank i will also meet its obligations. This represents the equity in the system when there are no bank defaults.

Assuming that if a counterparty defaults at maturity, bank i will revalue its interbank lending assets, cross-holding assets and interbank liabilities and then maturity equity is

$$\widetilde{E}_{i}\left(T\right) = \overline{A}^{c}_{i}\left(T\right) + \widetilde{A}^{f}_{i}\left(T\right) + \widetilde{A}^{s}_{i}\left(T\right) - \overline{L}^{c}_{i}\left(T\right) - \widetilde{L}^{f}_{i}\left(T\right). \tag{8}$$

Banks cannot reduce their payable amounts due to a decrease in their own assets, $\widetilde{L}^{f}_{i}\left(T\right) = \overline{L}^{f}_{i}\left(T\right)$. The value of the assessed interbank assets is obtained through the equilibrium clearing solution of Equation (**??**):

$$\widetilde{A}_i^f(T) = \sum_{j=1}^{N} \widetilde{A}_{ij}^f(T) = \sum_{j=1}^{N} f_{ij}\overline{L}_j^*, \tag{9}$$

$$\widetilde{A}_i^s(T) = \sum_{j=1}^{N} \widetilde{A}_{ij}^s(T) = \sum_{j=1}^{N} s_{ij}\overline{S}_j^*. \tag{10}$$

The valuation process for interbank lending assets and cross-holding assets is as follows:

$$\widetilde{A}_{ij}^f(T) = A_{ij}^f(T)V_j^f(\widetilde{E}_j(T)), \tag{11}$$

$$\widetilde{A}_{ij}^s(T) = A_{ij}^s(S)V_j^s(\widetilde{E}_j(T)), \tag{12}$$

thus, the valuation functions are

$$V_j^f\left(\widetilde{E}_j(T)\right) = \frac{\widetilde{A}_{ij}^f(T)}{\overline{A}_{ij}^f(T)} = \frac{\overline{L}_j^*}{\overline{L}_j} = \begin{cases} 1 & ,\widetilde{E}_j(T) \geq 0, \\ (1+\frac{\widetilde{E}_j(T)}{\overline{L}_j(T)})^+ & ,\widetilde{E}_j(T) < 0, \end{cases} \tag{13}$$

$$V_j^s\left(\widetilde{E}_j(T)\right) = \frac{\widetilde{A}_{ij}^s(T)}{\overline{A}_{ij}^s(T)} = \frac{\overline{S}_j^*}{\overline{S}_j} = \begin{cases} \frac{\widetilde{E}_j(T)}{\overline{S}_j} & ,\widetilde{E}_j(T) \geq 0, \\ 0 & ,\widetilde{E}_j(T) < 0. \end{cases} \tag{14}$$

$V_j^f\left(\widetilde{E}_j(T)\right)$ is the valuation function of interbank lending assets and $V_j^s\left(\widetilde{E}_j(T)\right)$ is the valuation of cross-holding assets. These valuation functions are also regarded as discount factors for the face value of interbank assets. In which, $y^+ = \max\{0, y\}$. Obviously, the valuation of assets by bank i depends on the equity of its counterparties.

Let $\lambda_j = 1 + \frac{\widetilde{E}_j(T)}{\overline{L}_j(T)}$and $\rho_j = \frac{\widetilde{E}_j(T)}{\overline{S}_j}$, where λ_j and ρ_j represent the recovery rates of bank j for its debt and equity.

2.4. Ex-ante valuation model

When valuation is performed at $t < T$, there is ex-ante uncertainty about external assets and external liabilities. To model the price changes of external assets, we assume that external assets follow a geometric Brownian motion, while external liabilities are non-stochastic processes. In this study, we only discuss the case of external asset independence, i.e., $d\overline{A}_i^c(s) = \mu_i\overline{A}_i^c(s)\,ds + \sigma_i\overline{A}_i^c(s)\,dW_i(s)$, for $\forall i \in \mathcal{N}$, $s \in [t, T]$. Assume that the market is complete and free of arbitrages, banks are risk-neutral investors, and the risk-free interest rate is r. All banks can calculate the discounted expected value of

$\widetilde{\mathbf{E}}\left(T\right)$ under the equivalent martingale measure $\mathcal{Q}$ at time t based on available information, i.e., $\widetilde{\mathbf{E}}\left(t\right) \equiv e^{-r(T-t)}E^{Q}\left[\widetilde{\mathbf{E}}\left(T\right)\middle|\overline{\mathbf{A}}^{\mathbf{c}}(t)\right]$.

Substituting Equation (**??**), we have:

$$\begin{aligned}\widetilde{E}_i\left(t\right) =& e^{-r(T-t)}E^{Q}\left[\overline{A}_i^c\left(T\right)+\widetilde{A}_i^f\left(T\right)+\widetilde{A}_i^s\left(T\right)-\bar{L}_i\left(T\right)\middle|\overline{\mathbf{A}}^{\mathbf{c}}\left(t\right)\right]\\ =& e^{-r(T-t)}E^{Q}\left[\overline{A}_i^c\left(T\right)+\sum_{j=1}^{N}A_{ij}^f(T)V_j^f(\widetilde{E}_j\left(T\right))+\sum_{j=1}^{N}A_{ij}^s(T)V_j^s(\widetilde{E}_j\left(T\right))-\bar{L}_i\left(T\right)\middle|\overline{\mathbf{A}}^{\mathbf{c}}(t)\right].\end{aligned} \tag{15}$$

The discounting of the expectations of external assets, external liabilities, interbank lending assets and cross-holding assets under the risk-free rate r and the equivalent martingale measure $\mathcal{Q}$ is shown in Equations (**??**)-(**??**):

$$e^{-r(T-t)}E^{Q}\left[\overline{A}_i^c\left(T\right)\middle|\overline{\mathbf{A}}^{\mathbf{c}}(t)\right]=\overline{A}_i^c\left(t\right), \tag{16}$$

$$e^{-r(T-t)}E^{Q}\left[\bar{L}_i\left(T\right)\middle|\overline{\mathbf{A}}^{\mathbf{c}}(t)\right]=e^{-r(T-t)}\bar{L}_i\left(T\right)=\bar{L}_i\left(t\right), \tag{17}$$

$$\begin{aligned}e^{-r(T-t)}E^{Q}\left[A_{ij}^f\left(T\right)V_j^f\left(\widetilde{E}_j\left(T\right)\right)\middle|\overline{\mathbf{A}}^{\mathbf{c}}\left(t\right)\right] =& e^{-r(T-t)}A_{ij}^f\left(T\right)E^{Q}\left[V_j^f\left(\widetilde{E}_j\left(T\right)\right)\middle|\overline{\mathbf{A}}^{\mathbf{c}}\left(t\right)\right]\\ =& A_{ij}^f\left(t\right)E^{Q}\left[V_j^f\left(\widetilde{E}_j\left(T\right)\right)\middle|\overline{\mathbf{A}}^{\mathbf{c}}\left(t\right)\right],\end{aligned} \tag{18}$$

$$\begin{aligned}e^{-r(T-t)}E^{Q}\left[A_{ij}^s\left(T\right)V_j^s\left(\widetilde{E}_j\left(T\right)\right)\middle|\overline{\mathbf{A}}^{\mathbf{c}}\left(t\right)\right] =& e^{-r(T-t)}A_{ij}^s\left(T\right)E^{Q}\left[V_j^s\left(\widetilde{E}_j\left(T\right)\right)\middle|\overline{\mathbf{A}}^{\mathbf{c}}\left(t\right)\right]\\ =& A_{ij}^s\left(t\right)E^{Q}\left[V_j^s\left(\widetilde{E}_j\left(T\right)\right)\middle|\overline{\mathbf{A}}^{\mathbf{c}}\left(t\right)\right].\end{aligned} \tag{19}$$

[**?**] and [**?**] have demonstrated that endogenous recovery rate models tend to overestimate equity values. We follow [**?**] and [**?**], assumes that the recovery rate λ_j is exogenous. Thus, according to Equation (**??**), the conditional expectation of the ex-ante valuation function of the interbank lending asset can be rewritten as:

$$\begin{aligned}E^{Q}\left[V_j^f\left(\widetilde{E}_j\left(T\right)\right)\middle|\overline{\mathbf{A}}^{\mathbf{c}}(t)\right] =& Q\left(\widetilde{E}_j\left(T\right)\geq 0\middle|\overline{\mathbf{A}}^{\mathbf{c}}\left(t\right)\right)+\lambda_j Q\left(\widetilde{E}_j\left(T\right)<0\middle|\overline{\mathbf{A}}^{\mathbf{c}}\left(t\right)\right)\\ =& \lambda_j+\left(1-\lambda_j\right)Q\left(\widetilde{E}_j\left(T\right)\geq 0\middle|\overline{\mathbf{A}}^{\mathbf{c}}\left(t\right)\right).\end{aligned} \tag{20}$$

The valuation function of cross-holding assets can be obtained by first calculating the cross-holding assets based on the equity ownership shares, and then computing it using Equation (??). Thus, the conditional expectation of the two valuation functions can be obtained by simply calculating the probability that the equity of bank j is positive at T, i.e., the survival rate of the bank at maturity.

The discussion above only considered whether the bank defaulted on the maturity date and did not consider the variation in the bank's equity prior to the maturity date. In reality, bank defaults are rare events. [?] argued that at any time before maturity, bank equity may breach the default threshold and thus become insolvent. Therefore, the time of a bank's default is when equity first falls to a deterministic or stochastic threshold. Based on the above analysis, we redefine the default time of bank j as:

$$\tau_j = \inf\{s \in [t,T] : \widetilde{E}_j(s) < 0\}, \tag{21}$$

that is, the time when the bank's equity valuation first becomes negative. Consistent with the valuation process above, if $\tau_j > T$, i.e., the valuation of bank j's equity is non-negative until maturity, then bank j does not default and the asset is valued normally. If $\tau_j \leq T$, indicating that bank j defaults on or before the maturity date T, then the bank's assets should be discounted based on the recovery function. In this case, we replace the conditional expectation of the ex-ante valuation function of interbank lending assets in Equation (??) with

$$\begin{aligned} E^Q\left[V_j^f\left(\widetilde{E}_j(T)\right)\middle|\overline{\mathbf{A}}^{\mathbf{c}}(t)\right] &= Q\left(\tau_j > T|\overline{\mathbf{A}}^{\mathbf{c}}(t)\right) + \lambda_j Q\left(\tau_j \leq T|\overline{\mathbf{A}}^{\mathbf{c}}(t)\right) \\ &= \lambda_j + (1-\lambda_j)Q\left(\tau_j > T|\overline{\mathbf{A}}^{\mathbf{c}}(t)\right). \end{aligned} \tag{22}$$

The solution of the survival probability $Q\left(\tau_j > T \middle| \overline{\mathbf{A}}^{\mathbf{c}}(t)\right)$ for bank j has been solved in the Black-Cox model. Following Equations (??) and Equations (??), $\widetilde{E}_j(s) = \overline{A}_j^c(s) + \sum_{k=1}^N \widetilde{A}_{jk}^f(s) + \sum_{k=1}^N \widetilde{A}_{jk}^s(s) - \bar{L}_j(T)$. In reality, details of interbank lending assets are not publicly available. When bank i evaluates its interbank lending assets and cross-holding assets, it is difficult to obtain specific assets data of counterparty bank j. Therefore, it is difficult to compute $\sum_{k=1}^N \widetilde{A}_{jk}^f(s)$ and $\sum_{k=1}^N \widetilde{A}_{jk}^s(s)$ directly, nor can we determine whether bank j will default. To make the valuation results more realistic, referring to [?], we make the approximation: $\sum_{k=1}^N \widetilde{A}_{jk}^f(s) + \sum_{k=1}^N \widetilde{A}_{jk}^s(s) \approx \sum_{k=1}^N \widetilde{A}_{jk}^f(t)e^{r(s-t)} + \sum_{k=1}^N \widetilde{A}_{jk}^s(t)e^{r(s-t)} = e^{r(s-t)}[\widetilde{E}_j(t) - \overline{A}_j^c(t) + \bar{L}_j(T)]$. This approximation gives $\widetilde{E}_j(s) \approx \widetilde{E}_j(t)e^{r(s-t)} + \overline{A}_j^c(s) - \overline{A}_j^c(t)e^{r(s-t)}$, and thus the bank's survival condition at time s is $\overline{A}_j^c(s) \geq [\overline{A}_j^c(t) - \widetilde{E}_j(t)]e^{r(s-t)}$. To be consistent with the form of the results from the Black-Cox model, we rewrite the survival condition as

$$\overline{A}_j^c(s) \geq [\overline{A}_j^c(t) - \widetilde{E}_j(t)]e^{r(T-t)}e^{-r(T-s)}. \tag{23}$$

The survival condition in the Black-Cox model is $V(s) \geq K$. We take $V(s) = \overline{A}_j^c(s)$, $K = C = [\overline{A}_j^c(t) - \widetilde{E}_j(t)]e^{r(T-t)}$, and $\gamma = r$, which leads to the formula for the survival probability in Equation (**??**):

$$Q\left(\tau_j > T|\overline{\mathbf{A}}^{\mathbf{c}}(t)\right) = \begin{cases} 0 & ,\widetilde{E}_j(t) \leq 0, \\ \mathcal{N}\left[\frac{\log\left(\frac{\overline{A}_j^c(t)}{\overline{A}_j^c(t)-\widetilde{E}_j(t)}\right)-\frac{\sigma_j^2(T-t)}{2}}{\sigma_j\sqrt{T-t}}\right] - & \\ \frac{\overline{A}_j^c(t)}{\overline{A}_j^c(t)-\widetilde{E}_j(t)}\mathcal{N}\left[\frac{\log\left(\frac{\overline{A}_j^c(t)-\widetilde{E}_j(t)}{\overline{A}_j^c(t)}\right)-\frac{\sigma_j^2(T-t)}{2}}{\sigma_j\sqrt{T-t}}\right] & ,0 < \widetilde{E}_j(t) < \overline{A}_j^c(t), \\ 1 & ,\widetilde{E}_j(t) \geq \overline{A}_j^c(t). \end{cases} \tag{24}$$

where $\mathcal{N}$ is the cumulative distribution of Gaussian random variables with mean 0 and variance 1.

Proposition 2.2. *The valuation function given by Equation (**??**) is feasible.*

The proof of Proposition **??** is in Appendix **??**.

2.5. Systemic risk and systemically important banks

The financial system is an interconnected whole. When an institution experiences a shock, the resulting losses include not only the direct losses caused by the initial shock but also the contagion losses due to interbank transactions within the system. If a bank in the system suffers a loss and gets into distress, the risk can be contagious to the whole system through the interbank connection channel, which is the systemic risk.

In this paper, we will simulate the losses caused by external asset shocks to banks through stress tests. Suppose there is an external shock $\overline{\mathbf{A}}^{\mathbf{c}}(t) \to \overline{\mathbf{A}}^{\mathbf{c}}(t) + \Delta\overline{\mathbf{A}}^{\mathbf{c}}$ and the bank loses $\Delta\overline{\mathbf{A}}^{\mathbf{c}}$ on its external assets. These losses are absorbed by the bank's equity, i.e., the external shock causes a direct equity loss $\Delta\mathbf{E}^{shock} = \Delta\overline{\mathbf{A}}^{\mathbf{c}}$. We denote by $\widetilde{\mathbf{E}}^{(0)}(t)$ the equity matrix of the system after an external shock but before risk contagion has occurred. Starting with

$\widetilde{\mathbf{E}}^{(0)}(t)$ the equity vector $\widetilde{\mathbf{E}}^{*}(t)$ is obtained by iterating Equation (**??**) to obtain the equity vector of the system when it reaches the clearing equilibrium, and then the loss of contagion due to risk contagion $\Delta\mathbf{E}^{cont} = \widetilde{\mathbf{E}}^{(0)}(t) - \widetilde{\mathbf{E}}^{*}(t)$. Overall, the total loss $\Delta\mathbf{E}^{total}$ due to exogenous shocks consists of two components: direct loss, and contagion loss, i.e.

$$\Delta\mathbf{E}^{total} = \Delta\mathbf{E}^{shock} + \Delta\mathbf{E}^{cont}. \tag{25}$$

Systemic risk can be measured by calculating the relative contagion loss $(\Delta\mathbf{E}^{cont}/\mathbf{E}^{pre-shock}(t))$. A bank's systemic importance reflects its ability to damage the whole financial system. Banks with higher systemic importance are more likely to cause greater losses to the system and pose a higher risk to the system compared to other banks. Our paper quantifies the marginal contribution of bank i to the total contagion losses by calculating the difference ϕ_i between the contagion losses of the subsystem excluding bank i, $\mathcal{B}\backslash\{i\}$, and the contagion losses of the entire system $\mathcal{B}$. The marginal contribution $\phi_i = \sum_i \Delta E_i^{cont}(\mathcal{B}\backslash\{i\}) - \sum_i \Delta E_i^{cont}(\mathcal{B})$. The construction process of the subsystem $\mathcal{B}\backslash\{i\}$ involves adjusting the interbank assets and liabilities of bank i. Specifically, all interbank lending assets and cross-holding assets of bank i are converted into corresponding external liabilities of the counterparties, and all interbank liabilities are converted into corresponding external assets of the counterparties. This construction process can be interpreted as if bank i never existed in the system or has already defaulted and been liquidated. Consequently, its counterparties can recover the corresponding interbank assets from bank i, which are then reinvested as external assets. The systemic importance of bank i, is shown in Equation (**??**):

$$s_i = \frac{\phi_i}{\sum_i \phi_i}. \tag{26}$$

3. Stress Test

3.1. Data and parameters

In this section, we construct a representative banking subsystem for stress testing to simulate the state of the system after a shock, using real data from the Chinese banking system from 2016-2022. To ensure the completeness of the data, the sample is selected from 24 listed commercial banks in China. The required bank asset and liability data and daily closing price data are obtained from the CSMAR database. The subsystem accounts for more than 75% of the total assets of the entire commercial banking system, which is representative. The selected sample banks are shown in the **??**.

We then construct the interbank lending exposure matrix using the minimum-density (MD) method [**?**]. MD method overcomes the shortcomings of the maximum-entropy method (ME) on the network fully connected and thus deviating from the actual, retains the important features of the original interbank market, and is widely used in the construction of interbank lending relationships. The specific construction process is shown in Equation (**??**).

$$
\begin{aligned}
\min_{\overline{A}_i^f} \ & c \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{1}_{\left[\overline{A}_{ij}^f > 0\right]}, \\
\text{s.t.} \ & \sum_{j=1}^{N} \overline{A}_{ij}^f = \overline{A}_i^f, \quad \forall i = 1, 2, \cdots, N, \\
& \sum_{i=1}^{N} \overline{A}_{ij}^f = \overline{L}_i^f, \quad \forall j = 1, 2, \cdots, N, \\
& \overline{A}_{ij}^f \geq 0, \quad \forall i, j.
\end{aligned}
\tag{27}
$$

where c is the fixed cost of establishing a lending relationship in the banking system. The MD method is actually a constrained optimization problem for the interbank lending matrix. Typically, total interbank assets and total interbank liabilities in the banking subsystem are not equal. Here we refer to [**?**] and select the sum of interbank lending assets to adjust for total interbank liabilities.

The parameters required to calculate the valuation function are shown in Equations (**??**) and (**??**), including the debt recovery rate, the forward-looking time horizon, and the volatility of external assets. The parameter values in our study refer to the researches of [**?**] and [**?**]. In our paper, the recovery rate is set to 0.6 and the forward-looking time horizon $T - t$ is set to 1 year, which is consistent with the time intervals between stress tests conducted by the Bank of England and the Federal Reserve. The evaluation of external asset volatility follows the standard method of the Merton model. It is assumed that the equity value, which is the same as the external asset value, follows a geometric Brownian motion, i.e., $d\widetilde{E}_j(s) = \mu_j^E \widetilde{E}_j(s)\, ds + \sigma_j^E \widetilde{E}_j(s)\, dW_j(s)$, $\forall j$. Given that the equity value of bank j depends on its external assets, we can derive the relationship between the equity volatility σ_j^E and the external asset volatility σ_j at time t using Itô's lemma. The general relationship can be expressed as:

$$\sigma_j \overline{A}_j^c(t) \frac{\partial \widetilde{E}_j(t)}{\partial \overline{A}_j^c(t)} = \sigma_j^E \widetilde{E}_j(t), \ \forall j. \tag{28}$$

From Equation (23), we have $\frac{\partial \widetilde{E}_j(t)}{\partial \overline{A}_i^c(t)} = 1$, therefore,

$$\sigma_j = \frac{\widetilde{E}_j(t)}{\overline{A}_j^c(t)} \sigma_j^E, \ \forall j, \tag{29}$$

where σ_j^E is the annualised volatility. The daily volatility is first calculated based on the daily closing price data, and the annualised volatility is obtained by multiplying the daily volatility by $\sqrt{252}$[1].

3.2. Network structure

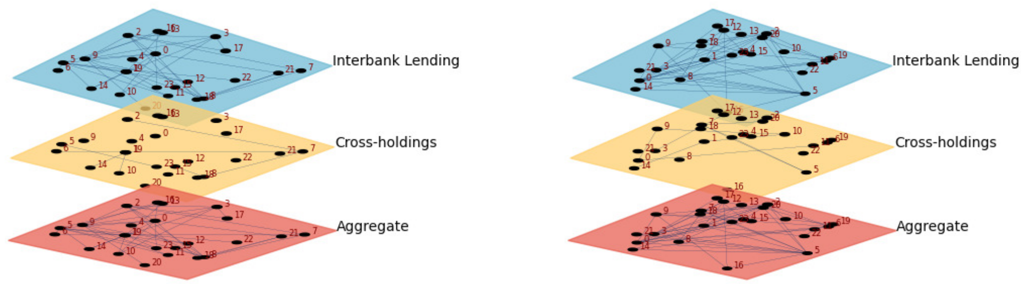
Fig. **??** illustrates a diagram of the two-layer banking network in 2016 and 2022. Comparing Fig. **??** and Fig. **??**, we find that compared to 2016, the number of two-layer network edges of Chinese banks in 2022 has increased, the network size is expanding, and the network connections tend to be more complex. Observing the nodes in the figure, some banks may be isolated nodes in one layer and have connected edges in other layers. These banks, as isolated nodes, are unable to contagion risk within layers, but are able to influence systemic equity by communicating risk through connections in other layers, thus enabling cross-layer contagion. The aggregation layer contains all the bank's connections in the two-layer network. Therefore, considering only a single-layer network is not sufficient to fully measure the risk of the banking system, resulting in an underestimation of risk. This shows that a single-layer network is not sufficient to fully measure the risk of the banking system.

3.3. Test results

3.3.1. Valuation result

Fig. **??** and Fig. **??** plot the valuation function as a function of equity for banks in 2016 and 2022. Overall, the Chinese banking system has grown

[1] Assuming there are 252 trading days in a year.



(a) 2016

(b) 2022

Figure 2: Comparison of two-layer networks in the banking system in 2016 and 2022

rapidly from 2016 to 2022, with a significant increase in net assets and an increase in system stability.

As can be seen in Fig. **??** and Fig. **??**, both the valuation function of interbank lending assets and the valuation function of cross-holding assets are non-decreasing curves with respect to equity. The left endpoint of the valuation function of interbank lending assets is 0.6, implying that when the bank's equity is 0, the counterparty can only recover interbank assets with a recovery rate of 0.6. The left endpoint of the valuation function of cross-holding assets is 0, implying that the counterparty will lose all of its cross-holding assets when bank's equity is 0. The right endpoint of the function curve corresponds to the valuation function at initial equity (unshocked). In 2022, the valuation functions of interbank lending assets and cross-holding assets for all banks are close to 1. This implies that there is almost no loss of all interbank lending assets of the system in the non-shock occurrence. In contrast, in 2016, there are some banks with the right endpoints of the valuation function of interbank lending assets and the valuation function of cross-holding assets much smaller than 1 (e.g. Bank16, Bank18, Bank21, Bank23, Bank24). This suggests that interbank lending assets would be at a discount even if no shock occurred. For example, for Bank 21, the right endpoint values of its interbank lending asset valuation function are 0.90, and the right endpoint value of its cross-holding asset valuation function is 0.75. This means that in the absence of shocks, its counterparties can only

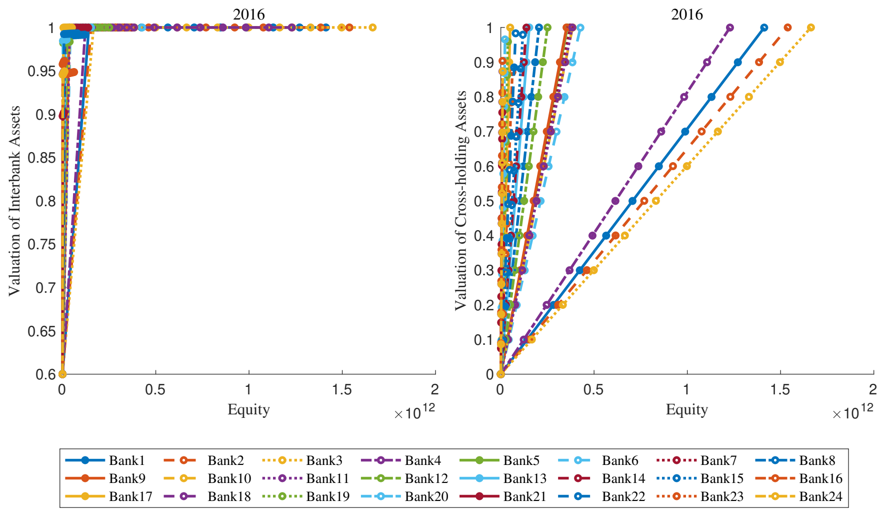


Figure 3: Valuation function in 2016

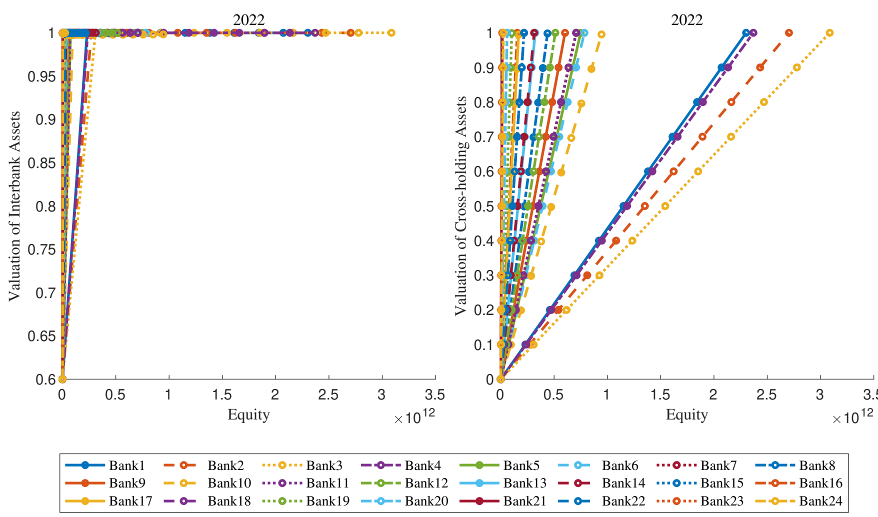


Figure 4: Valuation function in 2022

recover 90% of the interbank lending assets they hold and 75% of the cross-holding assets they hold, indicating a discount loss on interbank assets. This validates the need to study ex-ante contagion.

Both in 2016 and 2022, the larger banks have a less skewed function curve, i.e. they can withstand more shocks. In other words, for a given equity shock, the marginal change in the valuation function is greater for banks with smaller initial equity. In contrast, the valuation function for larger banks only starts to decline under very large shocks.

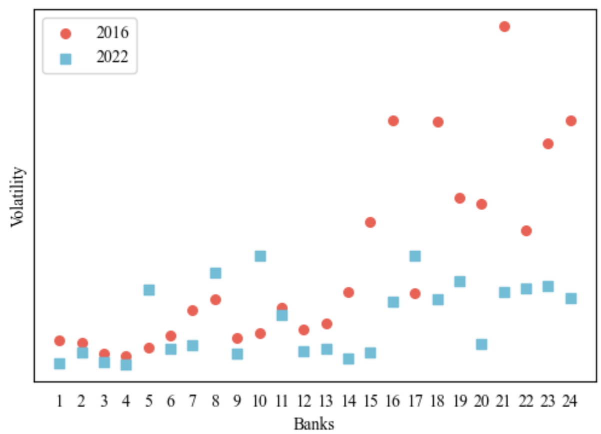


Figure 5: Comparison of annualized volatility in 2016 and 2022

Factors affecting the value of the valuation function are bank equity, default probability and recovery rate, and factors affecting the default rate are bank equity, external assets and asset volatility. All else being equal, higher asset volatility leads to a higher default probability, resulting in a lower value of the valuation function. Fig. **??** compares the annualized volatility of bank stock prices in 2016 and 2022. Compared to 2016, share price volatility is relatively low in 2022, and correspondingly, external asset volatility declines. In the absence of external shocks, both valuation functions converge to 1 for almost all banks in 2022.

3.3.2. Systemic risk

We simulate systematic scenarios at risk shocks of 0.2, 0.5 and 0.8 and briefly analyze the change in systematic risk for Chinese banks. Fig. **??** illustrates the systemic risk changes in 2016-2022. It can be seen that the trend of systemic risk changes in China fluctuates with the development of macroeconomic dynamics, but the systemic risk is significantly lower in 2019-2022 compared to 2016-2018.

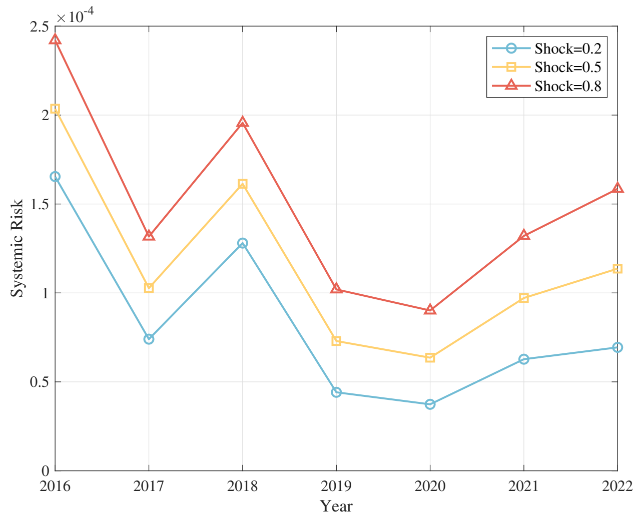


Figure 6: Changes in Systemic Risk of the Banking System from 2016 to 2022

Since 2015, the Chinese government has paid increasing attention to financial systemic risk, and under the strict supervision of the financial system, systemic risk generally showed a downward trend in 2016-2020, and the prevention and control of systemic risk in 2019-2020 achieved obvious results. During this period, China's financial systemic risk briefly rose in 2018 due to the impact of China's domestic "deleveraging" macroeconomic policies and international trade issues [**?**], weak domestic demand, external pressures, and continued volatility in China's capital markets [**?**]. Since 2021, structural problems such as delayed risk exposures have come to the fore, fueled by the new COVID-19 outbreak and severe government debt problems, exacerbating the creation of financial risks. Fortunately, the upward trend of systemic risks is weaker in 2021 and 2022 and remains manageable.

3.3.3. Systemically important banks

We simulate the state of banks at shocks of 0.5 and 0.9, respectively, and identify systemically important banks in China. Table **??** and Table **??** list the top 10 systemically important banks in 2016-2022 under the two shock states.Comparing systemically important banks under the two shocks, when the crisis had not yet occurred (Shock = 0.5), urban commercial banks appeared more frequently and ranked higher, thus reflecting their poorer risk resilience. When the crisis occurred (Shock = 0.9), the systemic importance was more related to the size of the bank's assets [?]. Therefore, regulatory authorities need to focus not only on "too big to fail" banks to prevent their failure from causing widespread risk contagion and potentially collapsing the entire banking system during [?], but also on smaller systemic importance banks during lesser impacts. It is crucial to prevent the accumulation of risks and mitigate the occurrence of larger crises.

Table 2: Top 10 Systemically Important Banks (Shock = 0.5)

Year	Bank
2016	Bank15, Bank1, Bank19, Bank4, Bank6, Bank2, Bank22, Bank16, Bank8, Bank24
2017	Bank2, Bank6, Bank1, Bank17, Bank4, Bank3, Bank5, Bank8, Bank14, Bank12
2018	Bank15, Bank2, Bank19, Bank4, Bank1, Bank3, Bank6, Bank10, Bank11, Bank24
2019	Bank2, Bank15, Bank20, Bank6, Bank24, Bank11, Bank1, Bank3, Bank8, Bank4
2020	Bank24, Bank2, Bank6, Bank17, Bank11, Bank1, Bank8, Bank5, Bank10, Bank19
2021	Bank2, Bank15, Bank8, Bank6, Bank1, Bank5, Bank10, Bank17, Bank11, Bank24
2022	Bank10, Bank2, Bank4, Bank13, Bank15, Bank17, Bank6, Bank3, Bank1, Bank24

Table 3: Top 10 Systemically Important Banks (Shock = 0.9)

Year	Bank
2016	Bank3, Bank2, Bank5, Bank4, Bank1, Bank15, Bank6, Bank8, Bank19, Bank22
2017	Bank2, Bank4, Bank3, Bank6, Bank1, Bank17, Bank5, Bank12, Bank10, Bank14
2018	Bank3, Bank4, Bank1, Bank15, Bank2, Bank5, Bank6, Bank10, Bank11, Bank19
2019	Bank3, Bank2, Bank1, Bank4, Bank8, Bank5, Bank6, Bank11, Bank14, Bank22
2020	Bank3, Bank2, Bank4, Bank1, Bank6, Bank5, Bank11, Bank8, Bank22, Bank17
2021	Bank3, Bank2, Bank6, Bank1, Bank4, Bank5, Bank8, Bank11, Bank17, Bank14
2022	Bank3, Bank2, Bank4, Bank6, Bank1, Bank5, Bank13, Bank15, Bank17, Bank14

4. Conclusion

In this study, we develop a two-channel ex-ante contagion model of systemic risk based on interbank lending network and cross-holding network from the perspective of two-layer network. Further, through stress tests we verify the necessity of ex-ante contagion and use real data to simulate systemic risk and systemically important banks in different shock scenarios. The main research and conclusions are as follows:

(1) We propose a two-channel systemic risk contagion mechanism based on interbank lending connections and cross-holding connections, and study

the two-layer network characteristics of the Chinese banking system. Real data suggests that the network size of China's banking system is expanding and network connections tend to be more complex. Cross-holding connections are relatively sparse. The two-layer network is able to tap more interbank relationships.

(2) We develop an ex-ante valuation model based on two-channel contagion. Based on the EN clearing framework and the Black-Cox model, the ex-ante contagion model proposed by [**?**] is extended to a two-layer network. Real data tests find that bankruptcy is not a necessary condition for systemic risk to arise, and it makes sense to explore ex-ante contagion mechanisms in the financial system. Stock price volatility affects the value of the valuation function.

(3) We measure China's systemic risk and identify systemically important banks through stress tests. Overall, China's systemic risk prevention and control policies have achieved some success, and changes in systemic risk are affected by the macroeconomic environment and the market environment. The simulation scenarios with different shock ratios differ in the systemically important banks identified by the model.

In this paper, we comprehensively consider the two connections between interbank lending and cross-holding to construct an ex-ante contagion model, which proves that bankruptcy is not a necessary condition for the generation of systemic risk, and that it is necessary to study the ex-ante contagion mechanism. This provides a reference framework for systemic risk assessment. Future extensions of this study may involve co-investments and fire sale, further research into more granular multi-layered networks of banks, and exploring the impact of indirect relationships of financial institutions on systemic risk contagion mechanisms.

CRediT authorship contribution statement

Yi Ding: Methodology, Formal analysis, Software, Writing - original draft. **Chun Yan**: Supervision, Writing - review & editing, Project administration. **Wei Liu**: Data curation, Project administration. **Man Qi**: Data curation. **Jiahui Liu**: Writing - review and editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work

reported in this paper.

Data availability

The datasets are available from the following source: CASMAR Database.

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Appendix A. Proof of Proposition ??

According to the fixed-point theorem, if a function f is a contraction mapping from $\mathbf{X}$ to $\mathbf{X}$, then $f(x)$ converges to a fixed point for all $x \in \mathbf{X}$. Therefore, it suffices to show that the function Φ is a contraction mapping.

Let $\mathbb{R}^N_{[0,\infty]} = \left\{(x_1, \ldots, x_N)^t, x_1, \ldots, x_N \in [0, \infty)\right\}$ denote the N-dimensional non-negative real space. In the metric space $\left(\mathbb{R}^N_{[0,\infty)}, d\right)$, we have

$$d\left(\mathbf{X}^h, \mathbf{X}^k\right) = \sum_{i=1}^{N} \left|X_i^h - X_i^k\right|, \ \mathbf{X}^h, \mathbf{X}^k \in \mathbb{R}^N_{[0,\infty)}. \tag{A.1}$$

The function Φ can be considered as a mapping from $\mathbb{R}^{2N}_{[0,\infty)}$ to $\mathbb{R}^{2N}_{[0,\infty)}$, where

$$d\left(\mathbf{Z}^h, \mathbf{Z}^k\right) = \sum_{i=1}^{N} \left[\left|L_i^{h*} - L_i^{k*}\right| - \left|S_i^{h*} - S_i^{k*}\right|\right], \ \mathbf{Z}^h, \mathbf{Z}^k \in \mathbb{R}^{2N}_{[0,\infty)}. \tag{A.2}$$

According to the definition of a contraction mapping, if there exists a real number λ such that

$$d\left(\Phi(\mathbf{Z}^h), \Phi(\mathbf{Z}^k)\right) \leq \lambda d\left(\mathbf{Z}^h, \mathbf{Z}^k\right), 0 \leq \lambda < 1, \tag{A.3}$$

for all $\left(\mathbf{Z}^h, \mathbf{Z}^k\right)$, then the function Φ is a contraction mapping from $\mathbf{Z}(T)$ to $\mathbf{Z}(T)$.

From Equation (**??**), we have:

$$
\begin{aligned}
& d\left(\Phi(\mathbf{Z}^h), \Phi(\mathbf{Z}^k)\right) \\
=& \sum_{i=1}^{N} \left\{ \left| \min\left[\overline{A}_i^c + \sum_{j=1}^{N}\left(f_{ij}\overline{L}_j^{h*} + s_{ij}\overline{S}_j^{\,h*}\right), \overline{L}_i\right] - \right.\right. \\
& \left. \min\left[\overline{A}_i^c + \sum_{j=1}^{N}\left(f_{ij}\overline{L}_j^{k*} + s_{ij}\overline{S}_j^{\,k*}\right), \overline{L}_i\right] \right| + \\
& \left| \max\left[\overline{A}_i^c + \sum_{j=1}^{N}\left(f_{ij}\overline{L}_j^{h*} + s_{ij}\overline{S}_j^{\,h*}\right) - \overline{L}_i, 0\right] - \right. \\
& \left.\left. \max\left[\overline{A}_i^c + \sum_{j=1}^{N}\left(f_{ij}\overline{L}_j^{k*} + s_{ij}\overline{S}_j^{\,k*}\right) - \overline{L}_i, 0\right] \right| \right\} \\
\le& \sum_{i=1}^{N} \left| \sum_{j=1}^{N} f_{ij}\left(\overline{L}_j^{h*} - \overline{L}_j^{k*}\right) + \sum_{j=1}^{N} s_{ij}\left(\overline{S}_j^{h*} - \overline{S}_j^{k*}\right) \right| \\
\le& \sum_{i=1}^{N} \left\{ \left(\sum_{j=1}^{N} f_{ij}\right)\left|\overline{L}_j^{h*} - \overline{L}_j^{k*}\right| + \left(\sum_{j=1}^{N} s_{ij}\right)\left|\overline{S}_j^{h*} - \overline{S}_j^{k*}\right| \right\} \\
=& \sum_{i=1}^{N} \left(f_j\left|\overline{L}_j^{h*} - \overline{L}_j^{k*}\right| + s_j\left|\overline{S}_j^{h*} - \overline{S}_j^{k*}\right| \right).
\end{aligned}
\tag{A.4}
$$

where the first inequality in the Equation (**??**) is derived from Inequality (**??**).

$$
\left|\min[x,b] - \min[x',b]\right| + \left|\max[x-b,0] - \max[x'-b,0]\right| \le |x - x'|\,,\ \forall x \ge 0, b \ge 0. \tag{A.5}
$$

From Remark **??**, we know that

$$
0 \le f_j \le 1,\ 0 \le s_j \le 1,\ \forall j \in \mathcal{N}, \tag{A.6}
$$

and there exists at least one j such that

$$
f_j \ne 1 \text{ or } s_j \ne 1. \tag{A.7}
$$

Therefore, there exists a real number λ such that

$$
d\left(\Phi(\mathbf{Z}^h), \Phi(\mathbf{Z}^k)\right) \le \lambda \sum_{i=1}^{N} \left(\left|\overline{L}_j^{\,h*} - \overline{L}_j^{\,k*}\right| + \left|\overline{S}_j^{\,h*} - \overline{S}_j^{\,k*}\right| \right) = \lambda d\left(\mathbf{Z}^h, \mathbf{Z}^k\right), 0 \le \lambda < 1, \tag{A.8}
$$

for all $\left(\mathbf{Z}^h, \mathbf{Z}^k\right)$. Thus, the function Φ is a contraction mapping from $\mathbb{R}^{2N}_{[0,\infty)}$ to $\mathbb{R}^{2N}_{[0,\infty)}$.

Appendix B. Proof of Proposition ??

According to Definition 2.1 and Theorem 3.1 of [?], if the range of the valuation function is $[0,1]$ and it is a non-decreasing upper continuous function, then the valuation function is feasible, and there exist maximal and minimal solutions.

Therefore, it is sufficient to prove: (1) the range of the valuation function is within $[0,1]$, (2) the valuation function is non-decreasing, (3) the valuation function is upper continuous.

(1) It is obviously that the probability $Q\left(\tau_j > T \middle| \overline{\mathbf{A}}^{\mathbf{c}}(t)\right)$ lies within the interval $[0,1]$. Therefore, the values of Equation (??) also lie within the interval $[0,1]$.

(2) It is sufficient to prove that Equation (??) is a non-decreasing function. Let $x = \frac{\overline{A}^c_j(t)}{\overline{A}^c_j(t)-\widetilde{E}_j(t)}$ and $y = \frac{\sigma_j^2(T-t)}{2}$. Rewriting Equation (??), we have

$$\begin{aligned} Q\left(\tau_j > T|\overline{\mathbf{A}}^{\mathbf{c}}(t)\right) &= \mathcal{N}\left[\frac{\log x - y}{\sqrt{2y}}\right] - x\mathcal{N}\left[-\frac{\log x + y}{\sqrt{2y}}\right] \\ &= \mathcal{N}\left[\frac{\log x - y}{\sqrt{2y}}\right] + x\mathcal{N}\left[\frac{\log x + y}{\sqrt{2y}}\right] - x, \end{aligned} \tag{B.1}$$

where $x > 1$ and $y > 0$. Taking the derivative of the conditional expectation with respect to $\widetilde{E}_j(t)$, we get:

$$\frac{\partial x}{\partial \widetilde{E}_j(t)}\left[\frac{1}{\sqrt{2y}x}\frac{e^{-\frac{1}{2}\left(\frac{\log x - y}{\sqrt{2y}}\right)^2}}{\sqrt{2\pi}} + \frac{1}{\sqrt{2y}}\frac{e^{-\frac{1}{2}\left(\frac{\log x + y}{\sqrt{2y}}\right)^2}}{\sqrt{2\pi}} + \mathcal{N}\left[\frac{\log x + y}{\sqrt{2y}}\right] - 1\right]. \tag{B.2}$$

Since $\frac{\partial x}{\partial \widetilde{E}_j(t)} = \frac{\overline{A}^c_j(t)}{\left(\overline{A}^c_j(t)-\widetilde{E}_j(t)\right)^2} > 0$ always holds, if

$$\frac{1}{\sqrt{2y}x}\frac{e^{-\frac{1}{2}\left(\frac{\log x - y}{\sqrt{2y}}\right)^2}}{\sqrt{2\pi}} + \frac{1}{\sqrt{2y}}\frac{e^{-\frac{1}{2}\left(\frac{\log x + y}{\sqrt{2y}}\right)^2}}{\sqrt{2\pi}} + \mathcal{N}\left[\frac{\log x + y}{\sqrt{2y}}\right] - 1 > 0, \tag{B.3}$$

then $Q\left(\tau_j > T \middle| \overline{\mathbf{A}}^{\mathbf{c}}(t)\right)$ is an increasing function with respect to $\widetilde{E}_j(t)$. Letting $z = \frac{1}{x}$, the above expression simplifies to

$$\frac{e^{-\frac{1}{2}\left(\frac{-\log z + y}{\sqrt{2y}}\right)^2}}{\sqrt{\pi y}} + \mathcal{N}\left[\frac{-\log z + y}{\sqrt{2y}}\right] - 1 > 0, \tag{B.4}$$

which holds for $0 < z < 1$.

(3) According to Equation (**??**), the valuation function may only be discontinuous at $\widetilde{E}_j(t) = 0$ and $\widetilde{E}_j(t) = \overline{A}_j^c(t)$. Since

$$\lim_{\widetilde{E}_j(t)\to 0^-} Q\left(\tau_j > T \mid \overline{\mathbf{A}}^{\mathbf{c}}(t)\right) = 0, \tag{B.5}$$

$$\begin{aligned}\lim_{\widetilde{E}_j(t)\to \overline{A}_j^c(t)^+} Q\left(\tau_j > T \mid \overline{\mathbf{A}}^{\mathbf{c}}(t)\right) &= \lim_{z\to 0^+} \mathcal{N}\left[\frac{-\log(z) - y}{\sqrt{2y}}\right] - \lim_{z\to 0^+} \frac{1}{z}\mathcal{N}\left[\frac{\log(z) - y}{\sqrt{2y}}\right] \\ &= 1 - \liminf_{z\to 0^+} \frac{1}{2\sqrt{\pi y z}} e^{-\frac{1}{4y}\log^2 z - \frac{y}{4}} = 1,\end{aligned} \tag{B.6}$$

thus, $Q\left(\tau_j > T \middle| \overline{\mathbf{A}}^{\mathbf{c}}(t)\right)$ is continuous at both $\widetilde{E}_j(t) = 0$ and $\widetilde{E}_j(t) = \overline{A}_j^c(t)$.

Therefore, the valuation function represented by Equation (**??**) is feasible.

Appendix C. Sample Banks

Table C.4: Sample banks

Label	Bank	Label	Bank
1	BANK OF CHINA	13	HUA XIA BANK
2	CHINA CONSTRUCTION BANK	14	BANK OF BEIJING
3	INDUSTRIAL AND COMMERCIAL BANK OF CHINA	15	BANK OF SHANGHAI
4	AGRICULTURAL BANK OF CHINA	16	BANK OF NANJING
5	INDUSTRIAL BANK	17	BANK OF NINGBO
6	BANK OF COMMUNICATIONS	18	WUXI RURAL COMMERCIAL BANK
7	SHANGHAI PUDONG DEVELOPMENT BANK	19	BANK OF HANGZHOU
8	PING AN BANK	20	BANK OF GUIYANG
9	CHINA MINSHENG BANK	21	SUZHOU RURAL COMMERCIAL BANK
10	CHINA MERCHANTS BANK	22	BANK OF JIANGSU
11	CHINA CITIC BANK	23	BANK OF JIANGYIN
12	CHINA EVERBRIGHT BANK	24	JIANGSU CHANGSHU RURAL COMMERCIAL BANK